A NOTE ON MAXIMAL SUBGROUPS OF FREE IDEMPOTENT GENERATED SEMIGROUPS OVER BANDS

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Abstract

We prove that all maximal subgroups of the free idempotent generated semigroup over a band B are free for all B belonging to a band variety $\mathbf{V}$ if and only if $\mathbf{V}$ consists either of left seminormal bands, or of right seminormal bands.

Let S be a semigroup, and let $E = E(S)$ be the set of its idempotents; in fact, E, along with the multiplication inherited from S, is a partial algebra. It turns out to be fruitful to restrict further the domain of the partial multiplication defined on E by considering only the pairs $e, f \in E$ for which either $ef \in \{e, f\}$ or $fe \in \{e, f\}$ (i.e. $\{ef, fe\} \cap \{e, f\} \neq \varnothing$). Note that if $ef \in \{e, f\}$ then fe is an idempotent, and the same is true if we interchange the roles of e and f. Such unordered pairs $\{e, f\}$ are called *basic pairs* and their products ef and fe are *basic products*.

The *free idempotent generated semigroup over E* is defined by the following presentation:

$$\mathsf{IG}(E) = \langle E \mid e \cdot f = ef \text{ such that } \{e, f\} \text{ is a basic pair} \rangle.$$

Here ef denotes the product of e and f in S (which is again an idempotent of S), while $\cdot$ stands for the concatenation operation in the free semigroup E^+ (also to be interpreted as the multiplication in its quotient $\mathsf{IG}(E)$). An important feature

Mathematics subject classification numbers: 20M05, 20M10, 20F05

Key words and phrases: free idempotent generated semigroup, band, maximal subgroup

The support of the Ministry of Education and Science of the Republic of Serbia, through Grant No. 174019, is gratefully acknowledged.

of $\mathsf{IG}(E)$ is that there is a natural homomorphism from $\mathsf{IG}(E)$ onto the subsemigroup of S generated by E, and the restriction of ϕ to the set of idempotents of $\mathsf{IG}(E)$ is a basic-product-preserving bijection onto E, see e.g. [5, 9, 13].

An important background to these definitions is the notion of the *biordered set* [7] of idempotents of a semigroup and its abstract counterpart. The biordered set of idempotents of S is just a partial algebra on $E(S)$ obtained by restricting the multiplication from S to basic pairs of idempotents. In this way we have that if B is a band (an idempotent semigroup), then, even though there is an everywhere defined multiplication on $E(B) = B$, its biordered set [3] is in general still a partial algebra. Another way of treating biordered sets is to consider them as relational structures $(E(S), \leqslant^{(l)}, \leqslant^{(r)})$, where the set of idempotents $E(S)$ is equipped by two quasi-order relations defined by

$$e \leqslant^{(l)} f \text{ if and only if } ef = e,$$
$$e \leqslant^{(r)} f \text{ if and only if } fe = e.$$

One of the main achievements of [4, 5, 9] is the result that the class of biordered sets considered as relational structures is *axiomatisable*: there is in fact a finite system of formulæ satisfied by biordered sets such that any set endowed with two quasi-orders satisfying the axioms in question is a biordered set of idempotents of some semigroup. In this sense we can speak about the free idempotent generated semigroup over a biordered set E. A fundamental fact which justifies the term ‘free’ is that $\mathsf{IG}(E)$ is the free object in the category of all semigroups S whose biordered set of idempotents is isomorphic to E: if $\psi : E \to E(S)$ is any isomorphism of biordered sets, then it uniquely extends (via the canonical injection of E into $\mathsf{IG}(E)$) to a homomorphism $\psi' : \mathsf{IG}(E) \to S$ whose image is the subsemigroup of S generated by $E(S)$. This is also true if ψ is a (surjective) homomorphism of biordered sets (taken as relational structures), so that the freeness property of $\mathsf{IG}(E)$ carries over to even wider categories of semigroups.

In this short note we consider $\mathsf{IG}(B)$, the free idempotent generated semigroup over (the biordered set of) a band B; more precisely, we are interested in the question whether the maximal subgroups of these semigroups are free. It was conjectured in [8] that each maximal subgroup of any semigroup of the form $\mathsf{IG}(E)$ is a free group. Recently, this was disproved [1] (see also [2]), where a certain 72-element semigroup was found whose biordered set E of idempotents yields a maximal subgroup in $\mathsf{IG}(E)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, the rank 2 free abelian group. Here we will see that a particular 20-element regular band suffices for the same purpose. In fact, as proved by Gray and Ruškuc in [6], *every* group can be isomorphic to a maximal subgroup of some $\mathsf{IG}(E)$, while the assumption that the semigroup S with $E = E(S)$ is finite yields a sole restriction that the groups

in question are finitely presented. This puts forward many new questions, one of which is the characterisation of bands B for which all subgroups of $\mathsf{IG}(B)$ are free.

More specifically, as a first approximation to the latter question, we may ask for a description of all varieties $\mathbf{V}$ of bands with the property that for each $B \in \mathbf{V}$ the maximal subgroups of $\mathsf{IG}(B)$ are free. To facilitate the discussion, we depict in Fig. 1 the bottom part of the lattice $\mathcal{L}(\mathbf{B})$ of all band varieties, along with their standard labels (see also [15, Diagram II.3.1]).

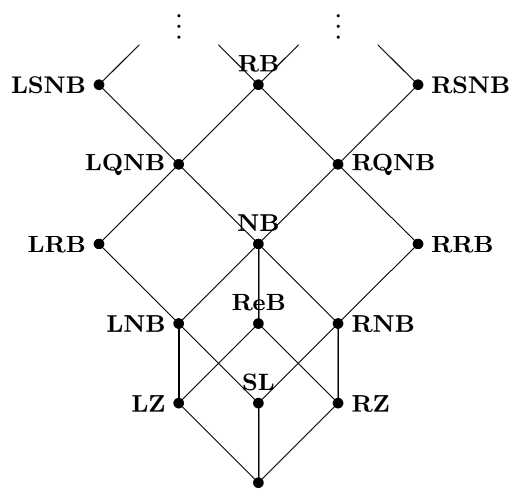

Figure 1: The bottom part of the lattice of all varieties of bands

The main result of this note is the following.

Theorem 1. *Let* $\mathbf{V}$ *be a variety of bands. Then* $\mathsf{IG}(B)$ *has all its maximal subgroups free for all* $B \in \mathbf{V}$ *if and only if* $\mathbf{V}$ *is contained either in* **LSNB** *or in* **RSNB**.

This theorem is a direct consequence of the following two propositions.

Proposition 2. *For any left (right) seminormal band* B*, all maximal subgroups of* $\mathsf{IG}(B)$ *are free.*

Proposition 3. *There exists a regular band* B *such that* $\mathsf{IG}(B)$ *has a maximal subgroup isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$.

The first of these propositions is a generalisation of the well known result of Pastijn [13, Theorem 6.5] (cf. also [10, 12]) that all maximal subgroups of

$\mathsf{IG}(B)$ are free for any normal band B. The other one supplies a simpler example with the same non-free maximal subgroup than the one considered in [1, Section 5]. The method used is the one from [6], which is based on the Reidemeister-Schreier type rewriting process for obtaining presentations of maximal subgroups of semigroups developed in [11]. So, before turning to the proofs of the above two propositions, we briefly present this general method yielding presentations for maximal subgroups of $\mathsf{IG}(E)$, $E = E(S)$, for an arbitrary semigroup S, and then we explain its particular case when S is a band. Along the way, we assume some familiarity with the most basic notions of semigroup theory, such as Green's relations and the structure of bands, see, for example, [7, 15].

Let S be a semigroup and let D be a $\mathcal{D}$-class of S containing an idempotent $e_0 \in E(S)$. We are going to label the $\mathcal{R}$-classes contained in D by R_i, $i \in I$, while L_j, $j \in J$, is the list of all $\mathcal{L}$-classes of D. The $\mathcal{H}$-class $R_i \cap L_j$ will be denoted by H_{ij}. Define $\mathcal{K} = \{(i,j) : H_{ij} \text{ is a group}\}$; as is well known, $(i,j) \in \mathcal{K}$ if and only if H_{ij} contains an idempotent, which we denote by e_{ij}. There is no loss of generality if we assume that both I and J contain an index 1, so that $e_0 = e_{11}$.

For a word $\mathbf{w} \in E^*$, let $\overline{\mathbf{w}}$ denote the image of $\mathbf{w}$ under the canonical monoid homomorphism of E^* into S^1: in other words, when $\mathbf{w}$ is non-empty, $\overline{\mathbf{w}}$ is just the element of S obtained by multiplying in S the idempotents the concatenation of which is $\mathbf{w}$. We say that a system of words $\mathbf{r}_j, \mathbf{r}'_j \in E^*$, $j \in J$, is a *Schreier system of representatives* for D if for each $j \in J$:

- the right multiplications by $\overline{\mathbf{r}_j}$ and $\overline{\mathbf{r}'_j}$ are mutually inverse $\mathcal{R}$-class preserving bijections $L_1 \to L_j$ and $L_j \to L_1$, respectively (so, in particular, right multiplication by $\mathbf{r}_1$ is the identity mapping on L_1);

- each prefix of $\mathbf{r}_j$ coincides with $\mathbf{r}_{j'}$ for some $j' \in J$ (in particular, the empty word is just $\mathbf{r}_1$).

It is well-known that such a Schreier system always exists. In the following, we assume that one particular Schreier system has been fixed.

In addition, we will assume that a mapping $i \mapsto j(i)$ has been specified such that $(i, j(i)) \in \mathcal{K}$: such $j(i)$ must exist for each $i \in I$, since D is a regular $\mathcal{D}$-class (as it contains an idempotent), and so each $\mathcal{R}$-class R_i must contain an idempotent. The index $j(i) \in J$ is called the *anchor* of R_i.

Finally, call a *square* a quadruple of idempotents (e, f, g, h) in D such that

$$\begin{array}{ccc} e & \mathcal{R} & f \\ \mathcal{L} & & \mathcal{L} \\ g & \mathcal{R} & h. \end{array}$$

Then there are $i, k \in I$ and $j, \ell \in J$ such that $e \in H_{ij}$, $f \in H_{i\ell}$, $g \in H_{kj}$ and $h \in H_{k\ell}$. For an idempotent $\varepsilon \in S$ we say that it *singularises* the square (e, f, g, h) if any of the following two cases takes place:

(a) $\varepsilon e = e$ and $\varepsilon g = g$, while $e = f\varepsilon$; or

(b) $e = \varepsilon g$, along with $e\varepsilon = e$ and $f\varepsilon = f$.

Note that case (a) implies $\varepsilon f = f$, $\varepsilon h = h$, $e\varepsilon = e$ and $g = g\varepsilon = h\varepsilon$, while conditions $\varepsilon e = e$, $f = \varepsilon f = \varepsilon h$, $g\varepsilon = g$ and $h\varepsilon = h$ follow from (b). The square (e, f, g, h) is *singular* if it is singularised by some idempotent of S. Let Σ be the set of all quadruples $(i, k; j, \ell) \in I \times I \times J \times J$ (to be called *singular rectangles*) such that $(e_{ij}, e_{i\ell}, e_{kj}, e_{k\ell})$ is a singular square in D.

The required general result of [6] can be now paraphrased as follows.

Theorem 4 (Theorem 5 of [6])**.** *Let S be a semigroup with a non-empty set of idempotents $E = E(S)$. With the notation as above, the maximal subgroup of the free idempotent generated semigroup $\mathsf{IG}(E)$ containing $e_{11} \in E$ is presented by $\langle \Gamma \mid \mathfrak{R} \rangle$, where $\Gamma = \{f_{ij} :\ (i,j) \in \mathcal{K}\}$, while $\mathfrak{R}$ consists of three types of relations:*

(i) $f_{i,j(i)} = 1$ *for all* $i \in I$*;*

(ii) $f_{ij} = f_{i\ell}$ *for all* $i \in I$ *and* $j, \ell \in J$ *such that* $\mathbf{r}_j \cdot e_{i\ell} = \mathbf{r}_\ell$*;*

(iii) $f_{ij}^{-1} f_{i\ell} = f_{kj}^{-1} f_{k\ell}$ *for all* $(i, k; j, \ell) \in \Sigma$.

For our purpose, we would like to focus on the particular case when S is a band. Then, clearly, $\mathcal{K} = I \times J$ and $D = \{e_{ij} :\ i \in I,\ j \in J\}$. Since $\mathcal{D} = \mathcal{J}$ in any band, the set of all $\mathcal{D}$-classes of B is partially ordered; it instantly turns out that, by definition, if ε singularises a square (e, f, g, h) in D, then $D_\varepsilon \geqslant D$. Now any such $\varepsilon \in B$ induces a pair of transformations on I and J, respectively, in the following sense. For each $i \in I$ and $j \in J$ there are $i', k \in I$ and $j', \ell \in J$ such that $\varepsilon e_{ij} = e_{i'\ell}$ and $e_{ij}\varepsilon = e_{kj'}$. One immediately sees that it must be $\ell = j$ and $k = i$, so that B acts on the left on I and on the right on J. Thus it is convenient to write the transformation $\sigma = \sigma_\varepsilon^{(l)}$ induced by ε on I to the left of its argument (so that $ee_{ij} = e_{\sigma(i)j}$), while the analogous transformation $\sigma' = \sigma_\varepsilon^{(r)}$ on J is written to the right (resulting in the rule $e_{ij}e = e_{i(j)\sigma'}$).

Corollary 5. *Let B be a band, let D be a $\mathcal{D}$-class of B, and let $e_{11} \in D$. Then the maximal subgroup $G_{e_{11}}$ of $\mathsf{IG}(B)$ containing e_{11} is presented by $\langle \Gamma \mid \mathfrak{R} \rangle$, where $\Gamma = \{f_{ij} :\ i \in I, j \in J\}$ and $\mathfrak{R}$ consists of relations*

$$f_{i1} = f_{1j} = f_{11} = 1 \tag{1}$$

for all $i \in I$ and $j \in J$, and

$$f_{ij}^{-1} f_{i\ell} = f_{kj}^{-1} f_{k\ell}, \tag{2}$$

where for some $\varepsilon \in B$ such that $D_\varepsilon \geqslant D$ the indices $i, k \in I$, $j, \ell \in J$ satisfy one of the following two conditions:

(a) $\sigma_\varepsilon^{(l)}(i) = i$, $\sigma_\varepsilon^{(l)}(k) = k$, *and* $(j)\sigma_\varepsilon^{(r)} = (\ell)\sigma_\varepsilon^{(r)} = \ell$,

(b) $\sigma_\varepsilon^{(l)}(i) = \sigma_\varepsilon^{(l)}(k) = k$, $(j)\sigma_\varepsilon^{(r)} = j$ *and* $(\ell)\sigma_\varepsilon^{(r)} = \ell$.

Proof. Since $\mathcal{K} = I \times J$, we have a generator f_{ij} for each $i \in I$ and $j \in J$. Furthermore, the same reason allows us to choose $j(i) = 1$ as the anchor for each $i \in I$. Such a choice will imply that the relations of type (i) from Theorem 4 take the form $f_{i1} = 1$, $i \in I$. In particular, we have $f_{11} = 1$. As for the Schreier system, we can choose $\mathbf{r}_1$ to be the empty word, $\mathbf{r}_j = e_{1j}$ for all $j \in J \setminus \{1\}$ and $\mathbf{r}'_j = e_{11}$ for all $j \in J$. The system $\mathbf{r}_j$, $j \in J$, of words over E is obviously prefix-closed. Since $e_{i1}e_{ij} = e_{ij}$ and $e_{ij}e_{11} = e_{i1}$ holds for all $i \in I$, $j \in J$, the right multiplications by e_{ij} and e_{11} are indeed mutually inverse bijections between L_1 and L_j and between L_j and L_1, respectively. Hence, the relations of type (ii) reduce to $f_{11} = f_{1j}$, that is, $f_{1j} = 1$, for all $j \in J$. Thus we have all the relations (1). Finally, the conditions (a) and (b) express precisely the singularisation of a square $(e_{ij}, e_{i\ell}, e_{kj}, e_{k\ell})$ in D by an element $\varepsilon \in B$; therefore, the relations (2) correspond to relations of type (iii). □

Rectangles $(i, k; j, \ell) \in I \times J$ of type (a) will be said to be *left-right* singular, while those of type (b) are *up-down* singular (with respect to ε). Another, more compact way of expressing condition (a) is $i, k \in \operatorname{Im} \sigma_\varepsilon^{(l)}$, $\ell \in \operatorname{Im} \sigma_\varepsilon^{(r)}$ and $(j, \ell) \in \operatorname{Ker} \sigma_\varepsilon^{(r)}$, while (b) is equivalent to $k \in \operatorname{Im} \sigma_\varepsilon^{(l)}$, $(i, k) \in \operatorname{Ker} \sigma_\varepsilon^{(l)}$ and $j, \ell \in \operatorname{Im} \sigma_\varepsilon^{(r)}$.

We can now turn to proving our aforementioned result.

Proof of Proposition 2. Without any loss of generality, assume that $B \in \mathbf{RSNB}$ (the case when B belongs to $\mathbf{LSNB}$ is dual). Recall (e.g. from [15, Proposition II.3.8]) that the variety $\mathbf{RSNB}$ satisfies (and is indeed defined by) the identity $tuv = tvtuv$. Therefore, if $B = \bigcup_{\alpha \in Y} B_\alpha$ is the greatest semilattice decomposition of B, $a \in B$ and $x, y \in D = B_\alpha$ for some $\alpha \in Y$, then $x = xyx$ and $y = yxy$. Hence, we have $ax = ax(yx) = ayxaxyx$ and $ay = ay(xy) = axyayxy$, implying $ax \,\mathcal{R}\, ay$. In particular, for any $\varepsilon \in B$ such that $D_\varepsilon \geqslant D$, $\varepsilon e_{ij} \,\mathcal{R}\, \varepsilon e_{k\ell}$ holds in D for all $i, k \in I$, $j, \ell \in J$, so the transformation $\sigma_\varepsilon^{(l)}$ is a constant function on I.

We conclude that there are no proper (non-degenerate) rectangles $(i, k; j, \ell)$ that are left-right singular with respect to some $\varepsilon \in B$. In other words, all proper

singular rectangles in $I \times J$—and thus all nontrivial relations of $G_{e_{11}}$—are of the up-down kind:

$$f_{ij}^{-1} f_{i\ell} = f_{k_0 j}^{-1} f_{k_0 \ell},$$

where j, ℓ are two fixed points of $\sigma_\varepsilon^{(r)}$, $i \in I$ is arbitrary, and (since in this context $\sigma_\varepsilon^{(l)}$ is constant) $\operatorname{Im} \sigma_\varepsilon^{(l)} = \{k_0\}$, for some $\varepsilon \in B$. However, now it is straightforward to deduce the relation (2) for *all* $i, k \in I$ and fixed points j, ℓ of $\sigma_\varepsilon^{(r)}$. Thus we are led to define an equivalence θ_B of $\bigcup_{\varepsilon \in B, D_\varepsilon \geqslant D} \operatorname{Im} \sigma_\varepsilon^{(r)} = J$ which is the transitive closure of the relation ρ_B defined by $(j_1, j_2) \in \rho_B$ if and only if $j_1, j_2 \in \operatorname{Im} \sigma_\varepsilon^{(r)}$ for some $\varepsilon \in B$. Now it is almost immediate to see that for all $i, k \in I$ and $j, \ell \in J$ such that $(j, \ell) \in \theta_B$ we have that

$$f_{ij}^{-1} f_{i\ell} = f_{kj}^{-1} f_{k\ell}$$

holds in $G_{e_{11}}$. This immediately implies $f_{k\ell} = 1$ for all $k \in I$ and $\ell \in 1/\theta_B$, as well as

$$f_{kj} = f_{k\ell}$$

for all $k \in I$, whenever $(j, \ell) \in \theta_B$. So, let $j_1 = 1, j_2 \dots, j_m \in J$ be a cross-section of J/θ_B. Then it is straightforward to eliminate all the relations from the presentation of $G_{e_{11}}$ while reducing its generating set to

$$\{f_{ij_r} : \ i \in I \setminus \{1\},\ 2 \leqslant r \leqslant m\}.$$

In other words, $G_{e_{11}}$ is a free group of rank $(|I| - 1)(m - 1)$. $\square$

Proof of Proposition 3. Let B be the subband of the free regular band on four generators a, b, c, d consisting of two $\mathcal{D}$-classes: a 2×2 class D_1 consisting of elements ab, aba, ba, bab and a 4×4 class D_0 consisting of elements of the form $\mathbf{u}_1 \mathbf{v} \mathbf{u}_2$, where $\mathbf{u}_1, \mathbf{u}_2 \in \{ab, ba\}$ and $\mathbf{v} \in \{cd, cdc, dc, dcd\}$. So, we can take $I = \{abcd, abdc, bacd, badc\}$, the set of all initial parts of words from D_0, and $J = \{cdba, dcba, cdab, dcab\}$, the set of all final parts of those words. A direct computation shows that

$$\sigma_{ab}^{(l)} = \sigma_{aba}^{(l)} = \begin{pmatrix} abcd & abdc & badc & bacd \\ abcd & abdc & abdc & abcd \end{pmatrix},$$

$$\sigma_{ba}^{(l)} = \sigma_{bab}^{(l)} = \begin{pmatrix} abcd & abdc & badc & bacd \\ bacd & badc & badc & bacd \end{pmatrix},$$

$$\sigma_{ab}^{(r)} = \sigma_{bab}^{(r)} = \begin{pmatrix} cdba & cdab & dcab & dcba \\ cdab & cdab & dcab & dcab \end{pmatrix},$$

$$\sigma_{ba}^{(r)} = \sigma_{aba}^{(r)} = \begin{pmatrix} cdba & cdab & dcab & dcba \\ cdba & cdba & dcba & dcba \end{pmatrix}.$$

If we enumerate (for brevity of further calculations) $abcd \to 1, abdc \to 2, badc \to 3, bacd \to 4$ and $cdba \to 1, cdab \to 2, dcab \to 3, dcba \to 4$, we get

$$\sigma^{(l)}_{ab} = \sigma^{(l)}_{aba} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \qquad \sigma^{(l)}_{ba} = \sigma^{(l)}_{bab} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix},$$
$$\sigma^{(r)}_{ab} = \sigma^{(r)}_{bab} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \qquad \sigma^{(r)}_{ba} = \sigma^{(r)}_{aba} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}.$$

Hence, the list of singular rectangles is exhausted by:

$$(1,2;1,2), (1,2;3,4), (3,4;1,2), (3,4;3,4),$$
$$(1,4;2,3), (1,4;1,4), (2,3;2,3), (2,3;1,4).$$

This results in $f_{11} = f_{12} = f_{13} = f_{14} = f_{21} = f_{31} = f_{41} = f_{22} = f_{44} = 1$ and

$$f_{23} = f_{24}, \quad f_{24} = f_{34}, \quad f_{43}^{-1} = f_{33}^{-1} f_{34}$$
$$f_{32} = f_{42}, \quad f_{42} = f_{43}, \quad f_{23} = f_{32}^{-1} f_{33}.$$

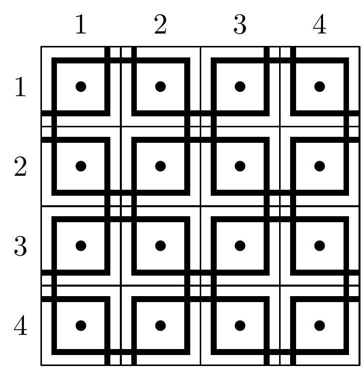

Figure 2: The rectangles in D_0 singularised by elements of D_1

If we denote $x = f_{23}$ and $y = f_{32}$ we obviously remain with these two generators for G_{abcdba} and a single relation

$$yx = f_{33} = xy,$$

so $G_{abcdba} \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

This completes the proof of Theorem 1.

Remark 6. The band B from the previous proof can be also realised as a regular subband of the free band FB_3 on three generators a, b, c whose elements are from $D'_1 = \{ab, aba, ba, bab\}$ and $D'_0 = \{\mathbf{u}c\mathbf{v} : \mathbf{u}, \mathbf{v} \in D'_1\}$.

We finish the note by several problems that might be subjects of future research in this direction.

Problem 1. Characterise all bands B with the property that $\mathsf{IG}(B)$ has a non-free maximal subgroup.

Problem 2. Characterise all groups that arise as maximal subgroups of $\mathsf{IG}(B)$ for some band B. The same problem stands for regular bands B, and in fact for $B \in \mathbf{V}$ for any particular band variety $\mathbf{V} \geqslant \mathbf{RB}$.

Problem 3. Given a band variety $\mathbf{V}$ and an integer $n \geqslant 1$, describe the maximal subgroups of $\mathsf{IG}(\mathfrak{F}_n\mathbf{V})$, where $\mathfrak{F}_n\mathbf{V}$ denotes the $\mathbf{V}$-free band on a set of n free generators [16].

Acknowledgement. The author is grateful to the anonymous referee, whose careful reading, comments and suggestions significantly improved the presentation of the results.

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